Computations of spheroidal harmonics with complex arguments: A review with an algorithm

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This paper not only reviews the various methodologies for evaluating the angular and radial prolate and oblate spheroidal functions and their eigenvalues, but also presents an efficient algorithm which is developed with the software package MATHEMATICA. Two algorithms are developed for computation of the eigenvalues λ_{mn} and coefficients d_r^{mn}. Important steps in programming are provided for estimating eigenvalues of the spheroidal harmonics with a complex argument c. Furthermore, the starting and ending points for searching for the eigenvalues by Newton's method are successfully obtained. As compared with the published data by Caldwell [J. Phys. A **21**, 3685 (1988)] or Press *et al.* [*Numerical Recipes in FORTRAN: The Art of Scientific Computing* (Cambridge University Press, Cambridge, 1992)] (for a real argument) and Oguchi [Radio Sci. **5**, 1207 (1970)] (for a complex argument), the spheroidal harmonics and their eigenvalues estimated using this algorithm are of a much higher accuracy. In particular, a lot of tabulated data for the intermediate coefficients $d_{\rho|r}^{mn}$, the prolate and oblate radial spheroidal functions of the second kind, and their first-order derivatives, as obtained by Flammer [*Spheroidal Wave Functions* (Stanford University Press, Stanford, CA, 1987)], are found to be inaccurate, although these tabulated data have been considered as exact referenced results for about half a century. The algorithm developed here for evaluating the spheroidal harmonics with the MATHEMATICA program is also found to be simple, fast, and numerically efficient, and of a much better accuracy than the other results tabulated by Flammer and others, being able to produce results of 100 significant digits or more.
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I. INTRODUCTION

Spheroidal harmonics are special functions in mathematical physics which have found many important and practical applications in science and engineering where the spheroidal coordinate system is used. In the evaluation of electromagnetic fields in spheroidal structures, spheroidal wave functions are frequently encountered, especially when the boundary-value problems in spheroidal structures are solved using full-wave analysis. By applying the separation of variables to Maxwell's equations of either the electric or magnetic field, two types of spheroidal harmonics, i.e., prolate and oblate functions corresponding to their respective coordinates system, can be obtained. Symbolically, one type of harmonics can be obtained from another simply by making the changes $c \to -ic$ and $\xi \to i\xi$. Computationally, the values of the prolate and oblate spheroidal wave functions are calculated in quite different ways.

In prolate (or oblate) spheroidal coordinates, the separation of the variables results in three independent functions: (1) the radial spheroidal function $R_{mn}^{(i)}(c,\xi)$ [or $R_{mn}^{(i)}(-ic,i\xi)$] $(i=1,2,3,4)$; (2) the angular spheroidal function $S_{mn}(c,\xi)$ [or $S_{mn}(-ic,\xi)$] [also referred to as the generalized Legendre function $\mathcal{P}_n^m(c,\xi)$]; and (3) the sine and cosine functions. The last pair of trigonometrical functions (sine and cosine) is well known, but the first two are not so easily computed. Computation of the spheroidal prolate (or oblate) radial (or angular) functions is involved in the eigenvalue computation and the forward and backward recursion formulation. Theoretically, the formulation of these harmonics was well documented by J. A. Stratton *et al.* earlier in 1956 [1] and Flammer in 1957 [2]. The computation of the eigenvalues [3–6] and the first-order derivatives [7–18] (of the angular and radial spheroidal wave functions) using FORTRAN and C programs have been very difficult.

Programs available to the public for numerically computing spheroidal harmonics and their eigenvalues are limited to the following: (i) a mathematical functions handbook recently published in 1992 by Baker, who provided many useful routines and codes in C language for computation of special functions including the spheroidal harmonics and their eigenvalues; (ii) a popular handbook series of routines and codes in BASIC, C, FORTRAN, and PASCAL (see Ref. [19]); (iii) two newly published handbooks, one by Zhang and Jin [20] and another by Thompson [21], in which the authors included a large number of FORTRAN programs that are capable of calculating a wide variety of special mathematical functions to a reasonable degree of accuracy. Other programs mentioned in the references are in general not directly accessible to the public, and therefore obtaining the source codes is not very convenient.

Basically, there are five methods available for evaluating the eigenvalues of spheroidal harmonics: (i) exact evaluation, on solving the transcendental equation in continued fraction form [1,2] or its equivalent [10]; (ii) an accurate evaluation by the relaxation method [3,8,19]; (iii) an approximate evaluation by power series expansion [5,14,17,18,22]; (iv) an approximate estimation by

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asymptotic expansions [6,22]; and (v) a systematic evaluation by casting the eigenvalue problem in a tridiagonal, symmetric matrix [5]. The results of the first method are very accurate, and have been used as referenced data for comparison [7,12,13]. The second method, developed by Caldwell [3], has become very popular recently, and has been implemented into the programs of both Refs. [19] and [8]; it converges quickly and gives reasonably good agreement between the evaluated results [3,8,19] and the exact results of Flammer [2], even when the value of c^2 becomes quite large. The third method has the advantage of rapid convergence when the value of c^2 is small, but its convergence becomes quite slow or the method may even fail when c^2 is large (e.g., $c \geqslant 10$) [5,6,14,17,18]. The fourth method provides simple and easy-to-use formulas for the evaluation of the eigenfunctions, but it is valid only for small c's. The fifth method, suggested by Hodge [5], reduces the eigenvalue problem to that of finding the eigenvalues of a real (or imaginary), tridiagonal symmetric matrix. Thus it allows for well-known procedures, which are rapid and accurate, to be used for the eigenvalue computation. It is more direct and systematic in comparison with other methods. Furthermore, it is considered to be a valuable tool when the computation of numerous eigenvalues is required.

Although there have been many published papers on computations of the eigenvalues of spheroidal harmonics over the past several decades, almost all of them are valid only for the real argument c. To the authors' knowledge, however, there are two exceptions [23]: one is the evaluation published by Oguchi in 1970 [6], and the other by Eglaya [4]. The former, using one infinite power series expansion for small c^2 and two asymptotic expansions for large c^2 to compute the complex eigenvalues, does not seem to be systematic and accurate enough, since the boundary for small and large values of c^2 is not clear $[\lim_{c\to 0}[\lambda_{mn}(c)] = n(n+1) + (c^2/2)\,(1-[(2m-1)(2m+1)/(2n-1)(2n+3)])$ and $\lim_{c\to\infty}[\lambda_{mn}(c)] = (2n-2m+1)c]$. The latter was published in Russian and thus has a limited readership, although it did compute the complex eigenvalues.

For a numerical computation of the spheroidal angular and radial functions, a similar situation exists, since computation of both functions involves the eigenvalues λ_{mn}. Several programs have been developed, as listed in the literature [7–18], but only a few of them are obtainable. Most importantly, the programs available in the literature for evaluation of the spheroidal harmonics are applicable only to real argument problems where the permittivity of each spheroidal region is lossless [i.e., $\mathrm{Im}(c)=0$ under prolate spheroidal coordinates, or $\mathrm{Re}(c)=0$ under oblate spheroidal coordinates]. Also, so far there is no commercial software such as MAPLE, MATHCAD, MATLAB, and MATHEMATICA available for computation of these spheroidal harmonics.

Nowadays, computer facilities are better, and there is a need to know which program or algorithm for computing these functions is the best with the current computer resources. In this connection, a comparison of the existing methods for computing spheroidal harmonics and their eigenvalues is made in this paper. Since MATHEMATICA software contains many symbolic and numerical built-in routines with simple commands or kernels, and is one of the most popular and mathematically powerful packages worldwide,

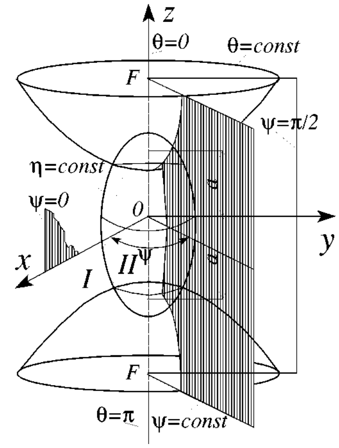

FIG. 1. Prolate spheroidal coordinates (ζ,θ,ψ).

an exact computation technique is adopted in this paper to evaluate the spheroidal harmonics and their eigenvalues. An efficient program routine consisting of functional commands of eigenvalues λ, angular $S_{mn}(c,\xi)$ and radial $R_{mn}^{(i)}(c,\xi)$ spheroidal harmonics, and their first-order derivatives has been developed. In particular, some important steps and initial values in numerical implementation, such as the Newton method for eigenvalues, are given. Both the prolate and oblate coordinate systems are considered. In the computation, there is no restriction on the dielectric properties (i.e., the medium can be either *lossless* or *lossy*) and the accuracy remains very high when the value of c^2 is very large. Although the routines for computing the eigenvalues and the spheroidal functions are not attached herein due to the length restriction, the complete software routine package will soon be available from the MATHSOURCE on the Web Page of Wolfram's MATHEMATICA.

II. SPHEROIDAL COORDINATES AND SPHEROIDAL HARMONICS

The prolate spheroidal coordinates shown in Fig. 1 are related to the rectangular coordinates by the following transformations:

$$x = \frac{d}{2}\sqrt{(1-\eta^2)(\xi^2-1)}\cos\,\phi, \tag{1a}$$

$$y = \frac{d}{2}\sqrt{(1-\eta^2)(\xi^2-1)}\sin\,\phi, \tag{1b}$$

$$z = \frac{d}{2}\,\eta\xi, \tag{1c}$$

with

$$-1 \leqslant \eta \leqslant 1, \quad 1 \leqslant \xi < \infty, \quad 0 \leqslant \phi \leqslant 2\pi, \tag{1d}$$

whereas the oblate spheroidal coordinates are related by

$$x = \frac{d}{2}\sqrt{(1-\eta^2)(\xi^2+1)}\cos\,\phi, \tag{2a}$$

$$y=\frac{d}{2}\sqrt{(1-\eta^2)(\xi^2+1)}\sin\,\phi, \tag{2b}$$

$$z=\frac{d}{2}\,\eta\xi, \tag{2c}$$

with

$$-1\leqslant\eta\leqslant 1,\quad 0\leqslant\xi<\infty,\quad 0\leqslant\phi\leqslant 2\pi \tag{2d}$$

or

$$0\leqslant\eta\leqslant 1,\quad -\infty<\xi<\infty,\quad 0\leqslant\phi\leqslant 2\pi. \tag{2e}$$

With these coordinate systems, the Helmholtz scalar wave equation becomes separable. The solutions of the wave equation are expressed by the scalar wave functions

$$\psi_{mn}=S_{mn}(c,\eta)R_{mn}(c,\xi)\frac{\cos}{\sin}m\phi \tag{3a}$$

for prolate spheroidal coordinates, and

$$\psi_{mn}=S_{mn}(-ic,\eta)R_{mn}(-ic,i\xi)\frac{\cos}{\sin}m\phi \tag{3b}$$

for oblate spheroidal coordinates, respectively. The four functions $S_{mn}(c,\eta)$, $R_{mn}(c,\xi)$, $S_{mn}(-ic,\eta)$, and $R_{mn}(-ic,i\xi)$, satisfy the following ordinary differential equations:

$$\frac{d}{d\eta}\left[(1-\eta^2)\frac{d}{d\eta}S_{mn}(c,\eta)\right]+\left[\lambda_{mn}-c^2\eta^2-\frac{m^2}{1-\eta^2}\right]\times S_{mn}(c,\eta)=0, \tag{4a}$$

$$\frac{d}{d\xi}\left[(\xi^2-1)\frac{d}{d\xi}R_{mn}(c,\xi)\right]-\left[\lambda_{mn}-c^2\xi^2+\frac{m^2}{\xi^2-1}\right]\times R_{mn}(c,\xi)=0 \tag{4b}$$

and

$$\frac{d}{d\eta}\left[(1-\eta^2)\frac{d}{d\eta}S_{mn}(-ic,\eta)\right]+\left[\lambda_{mn}+c^2\eta^2-\frac{m^2}{1-\eta^2}\right]\times S_{mn}(-ic,\eta)=0, \tag{5a}$$

$$\frac{d}{d\xi}\left[(\xi^2+1)\frac{d}{d\xi}R_{mn}(-ic,i\xi)\right]-\left[\lambda_{mn}-c^2\xi^2-\frac{m^2}{\xi^2+1}\right]\times R_{mn}(-ic,i\xi)=0. \tag{5b}$$

III. EIGENVALUES OF SPHEROIDAL HARMONICS OF COMPLEX ARGUMENT

A. Algorithm I using transcendental equation

The separation constants λ_{mn} in the above two pairs of equations are the eigenvalues of the *prolate* and *oblate* spheroidal *angular* and *radial* functions. These eigenvalues satisfy the transcendental equation

$$U_1(\lambda_{mn})+U_2(\lambda_{mn})=0, \tag{6}$$

where

$$U_1(\lambda_{mn})=\gamma^m_{n-m}-\lambda_{mn}-\frac{\beta^m_{n-m}}{\gamma^m_{n-m-2}-\lambda_{mn}-}\times\frac{\beta^m_{n-m-2}}{\gamma^m_{n-m-4}-\lambda_{mn}-}\quad\frac{\beta^m_{n-m-4}}{\gamma^m_{n-m-6}-\lambda_{mn}-}\cdots, \tag{7a}$$

$$U_2(\lambda_{mn})=-\frac{\beta^m_{n-m+2}}{\gamma^m_{n-m+2}-\lambda_{mn}-}\quad\frac{\beta^m_{n-m+4}}{\gamma^m_{n-m+4}-\lambda_{mn}-}\times\frac{\beta^m_{n-m+6}}{\gamma^m_{n-m+6}-\lambda_{mn}-}\cdots. \tag{7b}$$

In Eqs. (7a) and (7b), the compact notation

$$a-\frac{b}{c-}\quad\frac{d}{e-}\quad\frac{f}{g-}\quad\frac{h}{i-}\cdots$$

denotes the continued fraction

$$a-\cfrac{b}{c-\cfrac{d}{e-\cfrac{f}{g-\cfrac{h}{i-\cdots}}}},$$

and the intermediates β^m_r and γ^m_r are defined according to Flammer [2] by

$$\beta^m_r=\frac{r(r-1)(2m+r)(2m+r-1)}{(2m+2r-1)^2(2m+2r-3)(2m+2r+1)}c^4,\quad (r\geqslant 2) \tag{8a}$$

$$\gamma^m_r=(m+r)(m+r+1)+\frac{c^2}{2}\times\left[1-\frac{4m^2-1}{(2m+2r-1)(2m+2r+3)}\right]\quad (r\geqslant 0). \tag{8b}$$

As can be seen from the transcendental equation, the solution can be found by expanding the eigenvalues λ_{mn} into a Taylor series and then solving the polynomial equation, which was discussed by several published papers. The detailed expansion with a few given coefficients was addressed by Flammer [2]. However, the expansion is accurate only when the value of c^2 is not very large. When c^2 is very large, the Taylor series expansion technique fails because the series representation does not converge. In a similar fashion, the relaxation method [3,19] is also not accurate when c^2 is very large.

However, the solution for any c^2 can be found by solving the transcendental equation (6) directly. Newton's numerical technique can be employed to solve for its roots efficiently, where the estimated value of λ_{mn} and the starting and end points of the iterative technique are found to be

$$\lambda_{\text{estimate}}=n(n+1)+\text{Re}\left[\frac{c}{2}\right], \tag{9a}$$

$$\lambda_{\text{start}}=n(n+1)-c^2\left[1-\frac{(2m-1)(2m+1)}{(2n-1)(2n+3)}\right], \tag{9b}$$

$$\lambda_{\text{end}}=n(n+1)+c^2\left[1-\frac{(2m-1)(2m+1)}{(2n-1)(2n+3)}\right]. \tag{9c}$$

The values of $\lambda_{\text{estimate}}$, λ_{start}, and λ_{end} are tested in the program many times, and finally the above values are found. The routine for finding the roots λ_{mn} in Eq. (6) is available in Wolfram's web page.

As seen from the program, the routine consisting of simple statements is very compact. The advantage of the software MATHEMATICA has already been taken into consideration since the command ''Fold'' provides a simple and efficient implementation of the continued fraction.

To see if the program is efficient, the eigenvalues λ_{mn} of the spheroidal harmonics are computed. The results are compared with the data available elsewhere in the publications in Ref. [19] for real argument c, and in Ref. [6] for complex argument c.

For the real argument of c, as shown in Table I, the current routine produces exactly the same results as Flammer and is much better than the routine ''sfroid'' of the Numerical Recipe [19]. Also, the comparison between the currently computed results and Flammer's results shows an excellent agreement, but the current routine has the capability of producing results of 100 significant digits or more.

For the complex argument of c, as shown in Table II, this routine produces exact results [24]. The comparison also shows that Oguchi's results of eigenvalues for complex arguments are also good enough.

B. Algorithm II using recursive matrix equation

However, it should be pointed out that the above initial guesses in Eqs. (9a)–(9c) are only valid when c^2 is less than 1000. If this number is larger, other modifications must be made. This is because when the value of c is large, the required accuracy for λ_{start} in Eq. (9b) must be exceptionally high in order for Newton's method to evaluate the correct eigenvalues. Hence, the original formulation of λ_{start} is no longer accurate enough in the cases when $c^2>1000$. Under this condition, it is found that to solve for the eigenvalue by Newton's method is impractical. This is because there are many very closely spaced roots of Eq. (6) in a narrow range, so that a very accurate initial guess is necessary for computing the eigenvalue.

Thus, in the case of $c^2>1000$, the method proposed by Hodge [5] represents a more appropriate way of solving for the eigenvalues. A brief description of this method is given subsequently.

Substitution of Eq. (16a) into Eq. (4a) yields the following recurrence relation for the angular function expansion coefficients:

$$A_r^m(c)d_{r+2}^{mn}(c)+[B_r^m(c)-\lambda_{mn}(c)]d_r^{mn}(c)$$
$$+C_r^m(c)d_{r-2}^{mn}(c)=0, \quad r\geqslant 0, \tag{10}$$

where

$$A_r^m(c)=\frac{(2m+r+2)(2m+r+1)}{(2m+2r+3)(2m+2r+5)}c^2, \tag{11a}$$

$$B_r^m(c)=\frac{2(m+r)(m+r+1)-2m^2-1}{(2mm+2r-1)(2m+2r+3)}c^2$$
$$+(m+r)(m+r+1), \tag{11b}$$

$$C_r^m(c)=\frac{r(r-1)}{(2m+2r-3)(2m+2r-1)}c^2. \tag{11c}$$

Now, let

$$D_q=C_{2q+s}, \tag{12a}$$

$$E_q=B_{2q+s}, \tag{12b}$$

$$F_q=A_{2q+s}. \tag{12c}$$

TABLE I. Comparison of selected values of eigenvalue λ_{mn} computed by Flammer (1957), the Cambridge Numerical Recipe, and the present authors.

c^2	(m,n)	λ_{mn} Flammer (1957)	Numerical Recipe	This paper
−1.0	(4,11)	131.560	131.554	131.560 008 09
0.10	(2,2)	6.014 27	6.014 27	6.014 266 631 4
1.00	(1,1)	2.195 55	2.195 55	2.195 548 355
	(2,2)	6.140 95	6.140 95	6.140 948 992
	(2,5)	30.4361	30.4372	30.436 145 39
4.00	(1,1)	2.734 11	2.734 11	2.734 111 026
	(2,2)	6.542 50	6.542 53	6.542 495 274
16.0	(1,1)	4.399 59	4.399 61	4.399 593 067
	(2,5)	36.9963	37.0135	36.996 267 50

c is omitted from the above equations for simplicity. Thus the recurrence equation (10) becomes

$$D_q a_{q-1}+(E_q-\lambda_{mn})a_q+F_q a_{q+1}=0, \quad q\geqslant 0. \tag{13}$$

By a change of variables $a_q=(D_1D_2D_3\cdots D_q/F_0F_1\cdots F_{q-1})^{1/2}b_q$ in Eq. (13), and multiplying the resulting expression by $(F_0F_1\cdots F_{q-1}/D_1D_2D_3\cdots D_q)^{1/2}$, the following form of the recursion relation is obtained:

$$(D_qF_{q-1})^{1/2}b_{q-1}+(E_q-\lambda)b_q + (D_{q+1}F_q)^{1/2}b_{q+1}=0, \quad q\geqslant 0. \tag{14}$$

The above equation can be written in matrix form as follows:

$$\begin{bmatrix} (E_0-\lambda) & (D_1F_0)^{1/2} & 0 & 0 & \cdots \\ (D_1F_0)^{1/2} & (E_1-\lambda) & (D_2F_1)^{1/2} & 0 & \cdots \\ 0 & (D_2F_1)^{1/2} & (E_2-\lambda) & (D_3F_2)^{1/2} & \cdots \\ 0 & 0 & (D_3F_2)^{1/2} & (E_2-\lambda) & \cdots \\ . & . & . & . & \cdots \\ . & . & . & . & \cdots \\ . & . & . & . & \cdots \\ . & . & . & . & \cdots \end{bmatrix} \begin{bmatrix} b_0 \\ b_1 \\ b_2 \\ b_3 \\ . \\ . \\ . \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ . \\ . \\ . \end{bmatrix}. \tag{15}$$

When the matrix in Eq. (15) is obtained, the eigenvalue λ can then be evaluated directly and accurately using the command ‘‘Eigenvalue’’ in MATHEMATICA.

A compact program capable of fast computation of eigenvalues for large c has been developed. The results produced by the program agree well with the existing tabulated results in the literature [20,5,2]. As a sample of input-output, the eigenvalues have been computed for the cases where $m=1,2,3$; $n=3,4,\ldots,22,23$, and $c=0.5,1.0,2.5,5.0,10.0,25.0,50.0$, and 100.0. The results computed are compared with the values in Tables 15.1–15.4 of the handbook by Zhang [20]. It is found that the values are in good agreement. Like our previous routine of fractional calculus, this routine also has the advantage of producing results of 100 significant digits or more.

IV. CALCULATION OF ANGULAR SPHEROIDAL HARMONICS

To compute the angular spheroidal harmonics of the first and second kinds, i.e., $S_{mn}(c,\eta)$ and $S^{(2)}_{mn}(c,\eta)$, the following methods, namely, (i) the series expressions in terms of associated Legendre functions, and (ii) the power series representations, are normally used.

A. Series representation in terms of associated Legendre functions

The angular spheroidal function, as documented by Flammer [2], can be represented as a series of the associated Legendre functions of different orders. The relations between the angular spheroidal functions of the first and second kinds and the associated Legendre functions of different orders are given for the prolate and oblate spheroidal coordinates as follows: For the first kind,

$$S_{mn}(c,\eta)=\sum_{\ell=0,1}^{\infty}{}' \; d_{\ell}^{mn}(c)P_{m+\ell}^{m}(\eta), \tag{16a}$$

$$S_{mn}(-ic,\eta)=\sum_{\ell=0,1}^{\infty}{}' \; d_{\ell}^{mn}(-ic)P_{m+\ell}^{m}(\eta); \tag{16b}$$

TABLE II. Comparison of calculated eigenvalues λ_{mn} with corresponding values tabulated by Oguchi (1970).

c	(m,n)	Eigenvalues λ_{mn}	
		Oguchi (1970)	Computed
1.824 770+i2.601 670	(0,0)	1.705 180+i4.220 186	1.701 836+i4.219 998
2.094 267+i5.807 965	(0,2)	1.998 518+i8.578 716	1.993 901+i8.576 325
5.217 093+i3.081 362	(0,2)	23.915 82+i18.743 32	23.910 23+i18.741 94
3.563 644+i2.887 165	(0,1)	10.140 83+i11.121 58	10.137 05+i11.122 16
1.998 555+i4.097 453	(1,1)	2.915 318+i6.133 951	2.919 098+i6.134 851
3.862 833+i4.492 300	(1,2)	12.201 09+i16.244 07	12.196 91+i16.245 34
2.136 987+i5.449 457	(2,2)	6.102 540+i7.684 763	6.098 946+i7.684 379

and for the second kind,

$$
\begin{aligned}
S_{mn}^{(2)}(c,\eta) &= \sum_{\ell=-\infty}^{\infty}{}' \; d_{\ell}^{mn}(c) Q_{m+\ell}^{m}(\eta) \\
&= \sum_{\ell=-2m,-2m+1}^{\infty}{}' \; d_{\ell}^{mn}(c) Q_{m+\ell}^{m}(\eta) \\
&\quad + \sum_{\ell=2m+2,2m+1}^{\infty}{}' \; d_{\rho|\ell}^{mn}(c) P_{\ell-m-1}^{m}(\eta),
\end{aligned} \tag{17a}
$$

$$
\begin{aligned}
S_{mn}^{(2)}(-ic,\eta) &= \sum_{\ell=-\infty}^{\infty}{}' \; d_{\ell}^{mn}(-ic) Q_{m+\ell}^{m}(\eta) \\
&= \sum_{\ell=-2m,-2m+1}^{\infty}{}' \; d_{\ell}^{mn}(-ic) Q_{m+\ell}^{m}(\eta) \\
&\quad + \sum_{\ell=2m+2,2m+1}^{\infty}{}' \; d_{\rho|\ell}^{mn}(-ic) P_{\ell-m-1}^{m}(\eta),
\end{aligned} \tag{17b}
$$

where $P_n^m(x)$ is the associate Legendre function; herein and in the sequel, the prime over the summation sign indicates that the summation is over only even values of ℓ when $n-m$ is even, and over only odd values of ℓ when $n-m$ is odd; and

$$
d_{\rho|\ell}^{mn}(c) = \lim_{\rho\to 0} d_{-\ell+\rho}^{mn}/\rho. \tag{18}
$$

The intermediate parameter d_{ℓ}^{mn} is an important quantity frequently used in the formulation of the prolate (and oblate) angular (and radial) spheroidal harmonics of various kinds. Thus an independent section has been proposed so as to formulate and evaluate the parameter d_{ℓ}^{mn}. This issue will be addressed later.

The above series representation of the angular spheroidal harmonics are widely adopted [8–18] in the computation. The speed of convergence of the series is quite rapid, requiring only 4–14 terms (depending upon the value of c) to achieve necessarily good accuracy. Therefore, this paper actually developed an efficient algorithm based on these series expressions.

The angular spheroidal function of the second kind given above also serves as a special function in mathematical physics, but it has not found many applications in physics and electromagnetic wave theory. However, the angular spheroidal function of the first kind is an important special function which is commonly employed in physics and in the full-wave analysis of practical electromagnetic problems. Hence the angular function of the second kind will not be dealt with in great detail later in this paper.

B. Power series representation

Expanding the associated Legendre functions as a power series of $1-\eta^2$, we have the following expressions of the angular spheroidal harmonics for even and odd $(n-m)$:

$$
S_{mn}(c,\eta) = \begin{cases} (1-\eta^2)^{m/2} \displaystyle\sum_{k=0}^{\infty} c_{2k}^{mn}(c)(1-\eta^2)^k, & (n-m) \text{ even}, \\ \eta(1-\eta^2)^{m/2} \displaystyle\sum_{k=0}^{\infty} c_{2k}^{mn}(c)(1-\eta^2)^k, & (n-m) \text{ odd}, \end{cases} \tag{19a}
$$

and

$$
S_{mn}(-ic,\eta) = \begin{cases} (1-\eta^2)^{m/2} \displaystyle\sum_{k=0}^{\infty} c_{2k}^{mn}(-ic)(1-\eta^2)^k, & (n-m) \text{ even}, \\ \eta(1-\eta^2)^{m/2} \displaystyle\sum_{k=0}^{\infty} c_{2k}^{mn}(-ic)(1-\eta^2)^k, & (n-m) \text{ odd}, \end{cases} \tag{19b}
$$

where the coefficients c_{2k}^{mn} are related to d_r^{mn} by

$$
c_{2k}^{mn} = \frac{1}{2^m k!(m+k)!} \begin{cases} \displaystyle\sum_{r=k}^{\infty} \frac{(2m+2r)!}{(2r)!} \frac{\Gamma(k-r)}{\Gamma(-r)} \frac{\Gamma\left(k+m+r+\frac{1}{2}\right)}{\Gamma\left(m+r+\frac{1}{2}\right)} d_{2r}^{mn}, & (n-m) \text{ even}, \\ \displaystyle\sum_{r=k}^{\infty} \frac{(2m+2r+1)!}{(2r+1)!} \frac{\Gamma(k-r)}{\Gamma(-r)} \frac{\Gamma\left(k+m+r+\frac{3}{2}\right)}{\Gamma\left(m+r+\frac{3}{2}\right)} d_{2r+1}^{mn}, & (n-m) \text{ odd}. \end{cases} \tag{20}
$$

Inside Eq. (20), argument (c) is not specified. For both prolate (c) and oblate $(-ic)$ cases, Eq. (20) is always valid.

Besides the above power series of $1-\eta^2$, an alternative representation of the angular spheroidal harmonics in terms of the following simple ascending power series of η is given by

$$S_{mn}(c,\eta)=(1-\eta^2)^{m/2}\sum_{\ell=0,1}^{\infty}{}' \; p_{\ell}^{mn}(c)\,\eta^{\ell}, \tag{21a}$$

$$S_{mn}(-ic,\eta)=(-1)^{n-m}(1-\eta^2)^{m/2}e^{-c\eta}\sum_{\ell=0}^{\infty} B_{\ell}^{mn}(-ic)\times(1+\eta)^{\ell}=(1-\eta^2)^{m/2}e^{c\eta}\sum_{\ell=0}^{\infty} B_{\ell}^{mn}(-ic)(1-\eta)^{\ell}, \tag{21b}$$

where $p_{\ell}^{mn}(c)$ and $B_{\ell}^{mn}(-ic)$ satisfy respectively the recurrence relations as follows:

$$(\ell+1)(\ell+2)p_{\ell+2}^{mn}(c)-[\ell(\ell+2m+1)+m(m+1)-\lambda_{mn}(c)]p_{\ell}^{mn}(c)-c^2p_{\ell-2}^{mn}(c)=0, \tag{22a}$$

$$2(\ell+1)(\ell+m+1)B_{\ell+1}^{mn}(c)-[\ell(\ell+2m+1+4c)+(m+1)(m+2c)-\lambda_{mn}(-ic)-c^2]B_{\ell}^{mn}(-ic)+2(\ell+m)cB_{\ell-1}^{mn}(-ic)=0, \tag{22b}$$

and

$$\lim_{\ell\to\infty}\frac{p_{\ell+2}^{mn}(c)}{p_{\ell}^{mn}(c)}=0, \quad \text{and} \quad \lim_{\ell\to\infty}\frac{B_{\ell+1}^{mn}(-ic)}{B_{\ell}^{mn}(-ic)}=0. \tag{23}$$

These series forms in Eqs. (19a) and (19b) serve as alternative exact representations of the angular spheroidal harmonics where the associated Legendre functions are not readily available. The computational and convergence speeds for the evaluations of Eqs. (19a) and (19b) are almost the same as those of Eqs. (16a) and (16b). Equations (21a) and (21b) also give good results of the angular spheroidal functions, but the convergence of the series is not as rapid as that of the series of Eqs. (19a) and (19b) or (16a) and (16b).

C. Solution of auxiliary second-order differential equation

As was known previously, the prolate and oblate angular spheroidal harmonics satisfy the differential equation

$$\frac{d}{d\eta}\left[(1-\eta^2)\frac{d}{d\eta}S_{mn}(c,\eta)\right]+\left[\lambda_{mn}-c^2\eta^2-\frac{m^2}{1-\eta^2}\right]\times S_{mn}(c,\eta)=0. \tag{24}$$

To determine the solution of the spheroidal function uniquely, the boundary conditions given as follows may be used:

$$S_{mn}(c,0)=\begin{cases}\dfrac{(-1)^{n-m/2}(n+m)!}{2^n\left(\dfrac{n-m}{2}\right)!\left(\dfrac{n+m}{2}\right)!}, & (n-m)\ \text{even}\\[2ex] 0, & (n-m)\ \text{odd};\end{cases} \tag{25a}$$

$$S'_{mn}(c,0)=\begin{cases}0, & (n-m)\ \text{even},\\[1ex] \dfrac{(-1)^{n-m-1/2}(n+m+1)!}{2^n\left(\dfrac{n-m-1}{2}\right)!\left(\dfrac{n+m+1}{2}\right)!}, & (n-m)\ \text{odd}.\end{cases} \tag{25b}$$

The prolate angular spheroidal harmonics can also be obtained from the solution of the auxiliary differential equation and its boundary conditions in different forms, as stated in Refs. [9,10]. The only difference between this method and the method given by Abramov and co-workers is the boundary conditions utilized. Abramov and co-workers [9,10] applied the boundary and normalized conditions

$$S(c,0)=0,$$

where $S(c,1)$ is bounded and finite, and

$$\int_{-1}^{1}S_{mn}^2(c,\eta)=1,$$

or

$$S'(c,0)=0,$$

where $S(c,1)$ is bounded and finite, and

$$\int_{-1}^{1}S_{mn}^2(c,\eta)=1,$$

to solve numerically for the solution of the differential equations.

However, the boundary conditions given here are more numerically straightforward and efficient. Whatever the boundary conditions are, the solution of the differential equation is very computationally costful as compared with the

TABLE III. Comparison of calculated values of angular functions $S_{02}(3,\cos\theta)$, with corresponding values tabulated by Flammer (1957).

θ (in degrees)	$S_{02}(3.0,\cos\theta)$ Computed	Flammer (1957)
0	+1.040 93	+1.041
10	+1.023 06	+1.023
20	+0.963 95	+0.964
30	+0.849 71	+0.8497
50	+0.410 411	+0.4104
80	−0.417 055	−0.4171

methods mentioned above. So far, this method has been applied to the prolate spheroidal coordinates only. Further extension to the oblate spheroidal coordinates can be made as well.

D. Comparison of calculated results and Flammer's data

To show the accuracy of this program, we have calculated $S_{mn}(c)$ and obtained a great deal of data. In Table III, the currently calculated results of $S_{02}(3,\cos\theta)$ are compared with those tabulated by Flammer [2]. It is clear that very good agreement between the two sets of results are demonstrated, but the current results are of better accuracy to any degree of precision.

V. INTERMEDIATE COEFFICIENTS d_ℓ^{mn} AND $d_{\rho|\ell}^{mn}$

The coefficients, d_ℓ^{mn} used in many places for positive index $\ell \geqslant 0$ varying from 0 (or 1) to ∞ for even (or odd) $n-m$, and $d_{-k}^{mn}(c)$ with negative index $-k<0$ varying from $-2m$ (or $-2m+1$) to -2 (or -1) for even (or odd) $n-m$, are determined [2] from the continued fraction formulas

$$\frac{d_{\ell+2}^{mn}}{d_\ell^{mn}} = -\cfrac{q_{\ell+2}}{p_{\ell+2} - \cfrac{q_{\ell+4}}{p_{\ell+4} - \cfrac{q_{\ell+6}}{p_{\ell+6} - \ddots}}}, \quad \ell \geqslant 1, \tag{26a}$$

$$\frac{d_{-k}^{mn}}{d_{-k+2}^{mn}} = -\cfrac{\alpha_{-k}}{\beta_{-k} - \cfrac{\gamma_{-k}\alpha_{-k-2}}{\beta_{-k-2} - \cdots}}. \tag{26b}$$

It is found from the definition that

$$\frac{d_2^{mn}}{d_0^{mn}} = -p_0, \qquad \frac{d_3^{mn}}{d_1^{mn}} = -p_1,$$

$$p_\ell = \beta_\ell/\alpha_\ell, q_\ell = \gamma_\ell/\alpha_\ell.$$

The intermediates α_ℓ, β_ℓ, and γ_ℓ used above are given by

$$\alpha_\ell = \frac{(2m+\ell+2)(2m+\ell+1)}{(2m+2\ell+3)(2m+2\ell-1)} c^2, \tag{27a}$$

$$\beta_\ell = (m+\ell)(m+\ell+1) - \lambda_{mn}(c) + \frac{2(m+\ell)(m+\ell+1) - 2m^2 - 1}{(2m+2\ell-1)(2m+2\ell+3)} c^2, \tag{27b}$$

$$\gamma_\ell = \frac{\ell(\ell-1)}{(2m+2\ell-3)(2m+2\ell-1)} c^2, \tag{27c}$$

where $\lambda_{mn}(c)$'s denote the eigenvalues for a given c and assumed values m and n.

In order to determine the unique solutions of the d_ℓ^{mn}'s, the following two equations are also used: For even $n-m$,

$$\sum_{\ell=0}^{\infty}{}' \frac{(-1)^{\ell/2}(\ell+2m)!}{2^\ell\left(\frac{\ell}{2}\right)!\left(\frac{\ell+2m}{2}\right)!} d_\ell^{mn}(c) = \frac{(-1)^{(n-m)/2}(n+m)!}{2^{n-m}\left(\frac{n-m}{2}\right)!\left(\frac{n+m}{2}\right)!}, \tag{28a}$$

and for odd $n-m$,

$$\sum_{\ell=0}^{\infty}{}' \frac{(-1)^{(\ell-1)/2}(\ell+2m+1)!}{2^\ell\left(\frac{\ell-1}{2}\right)!\left(\frac{\ell+2m+1}{2}\right)!} d_\ell^{mn}(c) = \frac{(-1)^{(n-m-1)/2}(n+m+1)!}{2^{n-m}\left(\frac{n-m-1}{2}\right)!\left(\frac{n+m+1}{2}\right)!}. \tag{28b}$$

With the above equations, we can determine the coefficients $d_r^{mn}(c)$, where, most importantly, c can be a complex number and $r=-2m,\ldots,-2,0,2,\ldots$ for even $n-m$, or $-2m+1,\ldots,-1,1,3,\ldots$ for even $n-m$.

VI. CALCULATION OF RADIAL SPHEROIDAL HARMONICS

A. Series representation in terms of spherical Bessel functions

The radial spheroidal functions of the first to the fourth kinds can be expressed by

$$R_{mn}^{(i)}(c,\xi)=\frac{1}{\sum_{\ell=0,1}^{\infty} d_{\ell}^{mn}(c)\frac{(2m+\ell)!}{\ell!}}\left(\frac{\xi^2-1}{\xi^2}\right)^{m/2}\sideset{}{'}\sum_{\ell=0,1}^{\infty} i^{\ell+m-n}d_{\ell}^{mn}(c)\frac{(2m+\ell)!}{\ell!}z_{m+\ell}^{(i)}(c\xi), \tag{29a}$$

$$R_{mn}^{(i)}(-ic,i\xi)=\frac{1}{\sideset{}{'}\sum_{\ell=0,1}^{\infty} d_{\ell}^{mn}(-ic)\frac{(2m+\ell)!}{\ell!}}\left(\frac{\xi^2+1}{\xi^2}\right)^{m/2}\sideset{}{'}\sum_{\ell=0,1}^{\infty} i^{\ell+m-n}d_{\ell}^{mn}(-ic)\frac{(2m+\ell)!}{\ell!}z_{m+\ell}^{(i)}(c\xi), \tag{29b}$$

where $z_n^{(i)}(x)$ is the ith kind of spherical Bessel functions of order n, i.e., $z_n^{(1)}(x)=j_n(x)$, $z_n^{(2)}(x)=n_n(x)$, $z_n^{(3)}(x)=h_n^{(1)}(x)$, and $z_n^{(4)}(x)=h_n^{(2)}(x)$, respectively. Since $h_n^{(1)}(x)=j_n(x)+in_n(x)$ and $h_n^{(2)}(x)=j_n(x)-in_n(x)$, thus

$$R_{mn}^{(3)}(c,\xi)=R_{mn}^{(1)}(c,\xi)+iR_{mn}^{(2)}(c,\xi), \tag{30a}$$

$$R_{mn}^{(4)}(c,\xi)=R_{mn}^{(1)}(c,\xi)-iR_{mn}^{(2)}(c,\xi) \tag{30b}$$

for the prolate functions, and

$$R_{mn}^{(3)}(-ic,i\xi)=R_{mn}^{(1)}(-ic,i\xi)+iR_{mn}^{(2)}(-ic,i\xi), \tag{31a}$$

$$R_{mn}^{(4)}(-ic,i\xi)=R_{mn}^{(1)}(-ic,i\xi)-iR_{mn}^{(2)}(-ic,i\xi) \tag{31b}$$

for the oblate functions.

It is checked numerically that although the summation of Eq. (30a) is rapidly convergent, Eq. (30b) converges very slowly and similarly for Eqs. (31a) and (31b). It is observed from numerical evaluation that the spherical Bessel function of the second kind $n_{m+\ell}^{(i)}(c\xi)$ becomes larger and larger when its coefficient becomes smaller and smaller. However, their product varies quite slowly and remains almost constant when ℓ is large. Therefore, this set of equations (30b) and (31b) fails in convergence and is not recommended.

B. Proportional relations of angular and radial functions

Numerically, the radial functions $R_{mn}^{(1)}(c,\xi)$ and $R_{mn}^{(2)}(c,\xi)$ can be computed using the equations [2]

$$R_{mn}^{(1)}(c,\xi)=S_{mn}(c,\xi)/\kappa_{mn}^{(1)}(c), \tag{32a}$$

$$R_{mn}^{(1)}(-ic,i\xi)=S_{mn}(-ic,i\xi)/\kappa_{mn}^{(1)}(-ic), \tag{32b}$$

$$R_{mn}^{(2)}(c,\xi)=S_{mn}(c,\xi)/\kappa_{mn}^{(2)}(c), \tag{32c}$$

$$R_{mn}^{(2)}(-ic,i\xi)=S_{mn}(-ic,i\xi)/\kappa_{mn}^{(2)}(-ic) \tag{32d}$$

together with the coefficients $\kappa_{mn}^{(1)}$ and $\kappa_{mn}^{(2)}$ given by

$$\kappa_{mn}^{(1)}(c)=\frac{(2m+1)(n+m)!}{2^{n+m}d_0^{mn}(c)c^m m!\left(\frac{n-m}{2}\right)!\left(\frac{n+m}{2}\right)!}\times\sideset{}{'}\sum_{r=0}^{\infty} d_r^{mn}(c)\frac{(2m+r)!}{r!},(n-m)\quad\text{even}, \tag{33a}$$

$$\kappa_{mn}^{(1)}(c)=\frac{(2m+3)(n+m+1)!}{2^{n+m}d_1^{mn}(c)c^{m+1}m!\left(\frac{n-m-1}{2}\right)!}\frac{1}{\left(\frac{n+m+1}{2}\right)!}\times\sideset{}{'}\sum_{r=1}^{\infty} d_r^{mn}(c)\frac{(2m+r)!}{r!},\quad(n-m)\quad\text{odd} \tag{33b}$$

and

$$\kappa_{mn}^{(2)}(c)=\frac{2^{n-m}(2m)!\left(\frac{n-m}{2}\right)!\left(\frac{n+m}{2}\right)!d_{-2m}^{mn}(c)}{(2m-1)m!(n+m)!c^{m-1}}\times\sideset{}{'}\sum_{r=0}^{\infty} d_r^{mn}(c)\frac{(2m+r)!}{r!},\quad(n-m)\quad\text{even}, \tag{34a}$$

$$\kappa_{mn}^{(2)}(c)=-\frac{2^{n-m}(2m)!\left(\frac{n-m-1}{2}\right)!\left(\frac{n+m+1}{2}\right)!}{(2m-3)(2m-1)m!(n+m+1)!}\times\frac{d_{-2m+1}^{mn}(c)}{c^{m-2}}\sum_{r=1}^{\infty}{}^{IH}\, d_r^{mn}(c)\frac{(2m+r)!}{r!},\quad(n-m)\quad\text{odd}. \tag{34b}$$

It is found numerically that Eqs. (32a) and (32b) are easily computed if the prolate and oblate angular functions are obtained. Therefore, they are highly recommended. Also it is seen that in the computations, only the intermediate parameters $d_r^{mn}(c)$ and $d_r^{mn}(-ic)$ (where r starts from $-2m$ for even $n-m$ and from $-2m+1$ for odd $n-m$) are needed.

C. Power and Legendre functional series representations

1. Legendre functional series of radial functions of the second kind

Since $R_{mn}^{(1)}(c,\xi)$ and $R_{mn}^{(1)}(-ic,i\xi)$ can be obtained directly from the spherical Bessel functions $j_n(\xi)$, we will concentrate on the spheroidal harmonics of the second kind $R_{mn}^{(2)}(c,\xi)$ and $R_{mn}^{(2)}(-ic,i\xi)$ only. The following relations were derived by Flammer in 1957 [2]:

$$R_{mn}^{(2)}(c,\xi)=\frac{1}{\kappa_{mn}^{(2)}(c)}\left[\sum_{r=-2m,-2m+1}^{\infty}{}' d_r^{mn}(c)Q_{m+r}^m(\xi)+\sum_{r=2m+2,2m+1}^{\infty}{}' d_{\rho|r}^{mn}(c)P_{r-m-1}^m(\xi)\right], \tag{35a}$$

and likewise

$$R_{mn}^{(2)}(-ic,i\xi)=\frac{1}{\kappa_{mn}^{(2)}(-ic)}\left[\sum_{r=-2m,-2m+1}^{\infty}{}' d_r^{mn}(-ic)Q_{m+r}^m(i\xi)+\sum_{r=2m+2,2m+1}^{\infty}{}' d_{\rho|r}^{mn}(-ic)P_{r-m-1}^m(i\xi)\right]. \tag{35b}$$

In the above expressions, the defaulted setting of the Legendre type in numerical computations for $P_n^m(\xi)$ and $Q_n^m(\xi)$ should be changed from the real type (where there are branch cuts from $-\infty$ to -1 and $+1$ to $+\infty$) to the complex type (where there is a branch cut from $-\infty$ to $+1$). Otherwise, complex values of the associated Legendre functions $P_n^m(\xi)$ and $Q_n^m(\xi)$ $(\xi\geqslant 1)$ will be obtained, which is shown to be untrue. In versions 3.0 or 3.01 of MATHEMATICA, the defaulted definition of the associated Legendre function has been changed as compared with that in the previous version from 2.0 to 2.2.3. Corresponding changes should be made in accordance with the definitions by Flammer.

This method for computing the prolate and oblate radial spheroidal harmonics of the second kind is also a very good choice, since it provides an independent evaluation where only d_r^{mn} and the associated Legendre functions P_n^m and Q_n^m are needed.

2. Power series of radial functions of the first kind

Besides the above expression, another representation of the radial spheroidal functions is the series expression in terms of its powers of ξ^2-1 for prolate coordinates and of ξ^2+1 for oblate coordinates, given, respectively, by

$$R_{mn}^{(1)}(c,\xi)=\frac{(\xi^2-1)^{m/2}}{\kappa_{mn}^{(2)}(c)}\sum_{k=0}^{\infty}(-1)^k c_{2k}^{mn}(c)(\xi^2-1)^k \times\begin{cases}1, & (n-m)\text{ even}\\ \xi, & (n-m)\text{ odd,}\end{cases} \tag{36a}$$

$$R_{mn}^{(1)}(-ic,i\xi)=\frac{i^m(\xi^2+1)^{m/2}}{\kappa_{mn}^{(2)}(-ic)}\sum_{k=0}^{\infty} c_{2k}^{mn}(-ic)(\xi^2+1)^k \times\begin{cases}1, & (n-m)\text{ even}\\ i\xi, & (n-m)\text{ odd,}\end{cases} \tag{36b}$$

where c_{2k}^{mn} with real and imaginary arguments c and $-ic$ is, given, respectively, by Eq. (20) in terms of the coefficients $d_r^{mn}(c)$ and $d_r^{mn}(-ic)$, while $\kappa_{mn}^{(1)}(c)$ and $\kappa_{mn}^{(1)}(-ic)$ are given, respectively, by Eqs. (33a) and (33b).

As compared with Eqs. (29a) and (29b), the above expressions in Eqs. (36a) and (36b) are not so straightforward and require longer computation time. Therefore, they are not highly recommended in the numerical implementation.

3. Power series of radial functions of the second kind

The radial spheroidal harmonics of the second kind can also be expressed as follows:

$$R_{mn}^{(2)}(c,\xi)=\frac{1}{2}Q_{mn}(c)R_{mn}^{(1)}(c,\xi)\log\left(\frac{\xi+1}{\xi-1}\right)+g_{mn}(c,\xi), \tag{37a}$$

$$R_{mn}^{(2)}(-ic,i\xi)=Q_{mn}^{*}(-ic)R_{mn}^{(1)}(-ic,i\xi)\left(\log\xi-\frac{\pi}{2}\right)+g_{mn}(-ic,i\xi). \tag{37b}$$

The intermediate functions $Q_{mn}(c)$ and $Q_{mn}^{*}(-ic)$ are defined by

$$Q_{mn}(c)=\frac{[\kappa_{mn}^{(1)}(c)]^2}{c}\sum_{r=0}^{m} a_r^{mn}(c)\frac{(-1)^{m-r+1}}{r![2^{m-r}(m-r)!]^2}\begin{cases}(2m-2r)! & (n-m)\text{ even}\\ (2m-2r+1)! & (n-m)\text{ odd,}\end{cases} \tag{38a}$$

$$Q_{mn}^{*}(-ic)=(-1)^m\frac{[\kappa_{mn}^{(1)}(-ic)]^2}{c}\sum_{r=0}^{m}\frac{a_r^{mn}(-ic)}{r![2^{m-r}(m-r)!]^2}\begin{cases}(2m-2r)! & (n-m)\text{ even}\\ (2m-2r+1)! & (n-m)\text{ odd,}\end{cases} \tag{38b}$$

where

$$a_r^{mn}(\bullet)=\left\{\frac{d^r}{dx^r}\frac{1}{\left[\sum_{k=0}^{\infty}c_{2k}^{mn}(\bullet)x^k\right]^2}\right\}_{x=0}. \tag{39}$$

The other intermediate functions $g_{mn}(c)$ and $g_{mn}(-ic)$ are defined by

$$g_{mn}(c,\xi)=(\xi^2-1)^{-m/2}\sum_{r=0}^{\infty} b_r^{mn}(c)(\xi^2-1)^r\begin{cases}\xi, & (n-m)\text{ even}\\ 1, & (n-m)\text{ odd,}\end{cases} \tag{40a}$$

$$g_{mn}(-ic,i\xi)=(\xi^2+1)^{-m/2}\sum_{r=0}^{\infty} (-1)^{r-m/2}b_r^{mn}(-ic)(\xi^2+1)^r\begin{cases}i\xi, & (n-m)\text{ even}\\ 1, & (n-m)\text{ odd,}\end{cases} \tag{40b}$$

with

$$\begin{aligned}b_r^{mn}(\bullet)=&-\frac{1}{\kappa_{mn}^{(2)}(\bullet)}\Bigg\{\sum_{r=0}^{\infty} d_{2r}^{mn}(\bullet)\Bigg[\frac{r(r+m+1/2)(2m+2r-1)!}{2^{m-1}(m-1)!(2r+1)!}\\ &=-\frac{r(r+1)(r+m-1/2)(r+m+1/2)(2m+2r-2)!}{2^{m-1}(m-1)!(2r+1)!}+\sum_{k=0}^{[(m+2r-1)/2]}\frac{(2m+4r-4k-1)(2m+2r-2k-1)!}{2^m m!(2k+1)(m+2r-k)(2r-2k-1)!}\Bigg]\\ &-\sum_{k=m+1}^{\infty}\rho_{\rho|2r}^{mn}(\bullet)\frac{(2r-1)!}{2^m m!(2r-2m-1)!}\Bigg\},\quad (n-m)\text{ even,}\end{aligned} \tag{41a}$$

$$\begin{aligned}b_r^{mn}(\bullet)=&-\frac{1}{\kappa_{mn}^{(2)}(\bullet)}\Bigg\{\sum_{r=0}^{\infty} d_{2r+1}^{mn}(\bullet)\Bigg[\frac{(r+1)(r+m+1/2)(2m+2r)!}{2^{m-1}(m-1)!(2r+2)!}-\frac{(r+1)(r+m+1/2)[2m+r(r+m+3/2)]}{2^{m-1}(m-1)!}\\ &\times\frac{(2m+2r-1)!}{(2r+1)!}+\sum_{k=0}^{[(m+2r)/2]}\frac{(2m+4r-4k+1)(2m+2r-2k)!}{2^m m!(2k+1)(m+2r-k+1)(2r-2k)!}\Bigg]\\ &-\sum_{k=m+1}^{\infty}\rho_{\rho|2r-1}^{mn}(\bullet)\frac{(2r-2)!}{2^m m!(2r-2m-2)!}\Bigg\},\quad (n-m)\text{ odd,}\end{aligned} \tag{41b}$$

where the symbol ● stands for either c or $-ic$.

D. Representation using its differential equation

In a similar fashion to the angular spheroidal harmonics, the radial spheroidal functions of various kinds can also be obtained from the solution of the following differential equation:

$$\frac{d}{d\xi}\left[(\xi^2-1)\frac{d}{d\xi}R_{mn}^{(i)}(c,\xi)\right]-\left[\lambda_{mn}-c^2\xi^2+\frac{m^2}{\xi^2-1}\right]R_{mn}^{(i)}(c,\xi)=0. \tag{42}$$

To uniquely determine this, the boundary conditions must be utilized. Although some boundary conditions such as those in his equations (4.16.13)–(4.16.16b) were provided by Flammer [2], they are not recommended for use here since each of them consists of the parameters $d_r^{mn}(c)$ and $d_r^{mn}(-ic)$. If these parameters have been obtained, it is unnecessary to solve the equation numerically.

In Ref. [10] the conditions

$$|R_{mn}^{(1)}(c,1)|<\infty$$

and

$$R_{mn}^{(1)}(c,1)=\frac{1}{c\xi}\cos\left(c\xi-\frac{n+1}{2}\pi\right)+O(\xi^{-2}),\quad \xi\to\infty \tag{43a}$$

were suggested to determine $R_{mn}^{(1)}(c,\xi)$ uniquely and

$$R_{mn}^{(3)}(c,1)=\frac{1}{c\xi}\exp\left[i\left(c\xi-\frac{n+1}{2}\pi\right)\right]+O(\xi^{-2}),\quad \xi\to\infty \tag{43b}$$

to determine $R_{mn}^{(3)}(c,\xi)$ uniquely. The rest functions $R_{mn}^{(2)}(c,\xi)$ and $R_{mn}^{(4)}(c,\xi)$ can be obtained from Eqs. (30a) and (31b) or by using the forms of $1/c\xi\ \sin(c\xi-[(n+1)/2]\pi)$ and $1/c\xi\ \exp[i(c\xi-[(n+1)/2]\pi)]$ similar to the boundary conditions of Eqs. (33a) and (33b).

TABLE IV. Comparison of calculated values of the radial function of the first kind $R_{02}^{(1)}(3,\xi)$, with corresponding values tabulated by Flammer (1957).

ξ	$R_{02}^{(1)}(3,\xi)$ Computed	$R_{02}^{(1)}(3,\xi)$ Flammer (1957)
1.005	0.329 514	0.3295
1.020	0.334 628	0.3346
1.044	0.342 106	0.3421
1.077	0.350 931	0.3509

TABLE V. Comparison of selected values of $R_{mn}^{(2)}(c,\xi)$ computed by Flammer (1957), MacPhie and Do-Nhat, and the present authors.

c	(m,n)	ξ	$R_{mn}^{(2)}(c,\xi)$		
			Flammer	MacPhie and Do-Nhat	This paper
1.0	(2,2)	1.005	−375.0	−374.977 23	−374.977 22
2.0	(2,2)	1.005	−48.52	−48.522 271	−48.522 268
3.0	(2,3)	1.005	−37.45	−37.413 938	−37.428 719
4.0	(2,3)	1.005	−13.34	−13.331 298	−13.339 979

E. Numerical computation and comparison with Flammer's data

To show the efficiency of this program, we have calculated many data points of $R_{mn}^{(1)}(c,\xi)$ by using Eq. (29a). In Table IV, as an example, the calculated results of $R_{02}(3,\xi)$ are compared with those tabulated by Flammer [2]. A very good agreement between the two sets of results has been obtained. In a similar fashion, the current results have a better accuracy and higher precision.

VII. DERIVATIVES OF ANGULAR AND RADIAL SPHEROIDAL HARMONICS

It has been concluded that Eqs. (16a), (16b), (17a), (17b), (29a), (29b), (32c), and (32d) are desirable equations to compute $S_{mn}(c,\eta)$ and $S_{mn}(-ic,i\eta)$, $S_{mn}^{(2)}(c,\eta)$ and $S_{mn}^{(2)}(-ic,i\eta)$, $R_{mn}^{(1)}(c,\eta)$ and $R_{mn}^{(1)}(-ic,i\eta)$, and $R_{mn}^{(2)}(c,\eta)$ and $R_{mn}^{(2)}(-ic,i\eta)$. Therefore, their first-order derivatives can be obtained directly by taking the first-order derivatives of the associated Legendre functions and the spherical Bessel functions, respectively.

Although Eqs. (38a) and (38b) serve as the most straightforward formulas to compute prolate and oblate radial spheroidal functions of the second kind (as can be seen from the listed references), prolate and oblate radial spheroidal harmonics of the second kind can also be obtained from Eqs. (32c) and (32d). Also the following functional forms of the Wronskian test values of the radial functions of the first and second kinds are also frequently employed to compute $R_{mn}^{(2)}(c,\eta)$ and $R_{mn}^{(2)}(-ic,i\eta)$:

$$R_{mn}^{(1)}(c,\eta)\frac{d}{d\xi}R_{mn}^{(2)}(c,\eta)-R_{mn}^{(2)}(c,\eta)\frac{d}{d\xi}R_{mn}^{(1)}(c,\eta) = \frac{1}{c(\xi^2-1)}, \tag{44a}$$

and

$$R_{mn}^{(1)}(-ic,i\eta)\frac{d}{d\xi}R_{mn}^{(2)}(-ic,i\eta)-R_{mn}^{(2)}(-ic,i\eta) \times\frac{d}{d\xi}R_{mn}^{(1)}(-ic,i\eta)=\frac{1}{c(\xi^2+1)}. \tag{44b}$$

VIII. NUMERICAL COMPUTATION AND COMPARISON OF $R_{mn}^{(2)}(C,\xi)$ CALCULATED WITH MORE RECENT DATA

In a recent paper, the accuracy of the values of $R_{mn}^{(2)}(c,\xi)$ provided by Flammer [2], and again reproduced in Abramowitz's handbook [22], was argued by MacPhie and Do-Nhat [25]. It was claimed that the values given by Flammer were inaccurate.

MacPhie and Do-Nhat [25] then recalculated $R_{mn}^{(1)}(c,\xi)$ and $R_{mn}^{(2)}(c,\xi)$ using double precision, and a slightly different expansion series in which the functions $Q_{r+m}^{m}(\xi)$ and P_{r-m-1}^{m} in Eq. (35a) are expanded around $\xi=1$. Readers who are interested in this topic can refer to Ref. [25] or [26] for more details about the alternate expression. In view of this clarification, the set of values printed in Ref. [25] was recomputed with this program for verification purposes. An example table that compared the results from Flammer, MacPhie and Do-Nhat, and the authors are tabulated in Tables V and VI.

A comparison shows that for certain sets of m,n,c, and ξ, all the three sources produced different numerical solutions. This discrepancy prompted us to make a comparison of the computational accuracy for the three sets of solutions by computing the respective Wronskian values and comparing them against the theoretical Wronskian values.

It is observed that the computed Wronskian values using Flammer's result, in general, are lower in accuracy when

TABLE VI. Comparison of selected values of $R_{mn}^{(2)'}(c,\xi)$ computed by Flammer (1957), MacPhie and Do-Nhat, and the present authors.

c	(m,n)	ξ	$R_{mn}^{(2)}(c,\xi)$		
			Flammer	MacPhie and Do-Nhat	This paper
1.0	(2,2)	1.005	75 740	75 736.490	75 736.490
2.0	(2,2)	1.005	9738.0	9736.9853	9736.9859
3.0	(2,3)	1.005	7556.0	7569.0142	7566.0512
4.0	(2,3)	1.005	2662.0	1232.5894	2662.5329

TABLE VII. Computed Wronskian values: Comparison of results by Flammer (1957), MacPhie and Do-Nhat, and the present authors.

c	(m,n)	ξ	Wronskian values			
			Flammer	MacPhie and Do-Nhat	This paper	Theoretical
1.0	(2,2)	1.005	99.7560	99.7506	99.7506	99.7506
2.0	(2,2)	1.005	49.8768	49.8753	49.8753	49.8753
3.0	(2,3)	1.005	33.2374	33.2502	33.2502	33.2502
4.0	(2,3)	1.005	24.9352	18.2335	24.9377	24.9377

compared with the other two. The difference from the theoretical Wronskian value increases when the value of c increases. Thus MacPhie and Do-Nhat's claim was verified. Their values achieve a double precision accuracy for small value of c. However, their value decreases in accuracy as the value of c becames larger. One example is shown in Tables VI and VII where $n=3$, $m=2$, $\xi=1.005$, and $c=4.0$. The Wronskian value computed from the authors' values of $R_{mn}^{(2)}(c,\xi)$ and $R_{mn}^{(2)\prime}(c,\xi)$ is generally more accurate than MacPhie and Do-Nhat's computation and, to date, this is the most accurate computation of $R_{mn}^{(2)}(c,\xi)$ for arbitrary values of c known in the literature. No decrease in accuracy is observed when the value of c increases.

It should be noted that for more accurate values of $R_{mn}^{(2)}(c,\xi)$ (especially for the ranges $c=1,2,3$, and 4, $m\leqslant 2$, and $n\leqslant 2$), one should refer to the printed tables published by MacPhie and Do-Nhat [25], instead of the tables given by Flammer [2]. However, the values of $R_{mn}^{(2)}(c,\xi)$ tabulated by MacPhie and Do-Nhat in the ranges $m=2$, $n=3$, and $c=3,4$, and 5 are found to be rather inaccurate. This conclusion was arrived at after the Wronskian values computed using MacPhie and Do-Nhat's values of $R_{mn}^{(2)}(c,\xi)$ and $R_{mn}^{(2)\prime}(c,\xi)$ were found to differ from the theoretical Wronskian value by quite a large margin.

IX. ACCURATE VALUES OF OBLATE SPHEROIDAL RADIAL FUNCTIONS OF THE SECOND KIND

The claim by MacPhie and Do-Nhat [25] about the inaccuracy of Flammer's tabulated values of $R_{mn}^{(2)}(c,\xi)$ prompted the present authors to recalculate the oblate radial function of the second kind, i.e., $R_{mn}^{(2)}(-ic,i\xi)$. A detailed study of Flammer's work led us to discover some errors in the former's expression of the special values of $R_{mn}^{(2)}(-ic,i0)$. Numerical solutions show that his equations (4.6.14) and (4.6.15b) are incorrect. So we examined the equation theoretically. Using Eq. (32b), for odd $(n-m)$ we can express

$$R_{mn}^{(1)\prime}(-ic,i0)=\frac{S_{mn}(-ic,i0)}{\kappa_{mn}^{(1)}(0)}$$
$$=\frac{1}{(2m+3)(n+m+1)!\sum_{r-1}^{\infty}{}' \; d_r^{mn}(-ic)\dfrac{(2m+r)!}{r!}}$$
$$\times\left[2^{n+m}d_1^{mn}(-ic)(-ic)^{m+1}m!\left(\frac{n-m-1}{2}\right)!\left(\frac{n+m+1}{2}\right)!S_{mn}(-ic,0)\right]. \tag{45}$$

Then by using Eq. (45) and the special values in Eq. (25b), we can, in a straightforward manner, simplify $R_{mn}^{(1)}(-ic,0)$ to

$$R_{mn}^{(1)\prime}(-ic,i0)=\frac{i^{n-m-1}2^m m!c^{m+1}d_1^{mn}(-ic)}{(2m+3)\sum_{r=1}^{\infty} d_r^{mn}(-ic)\dfrac{(2m+r)!}{r!}}. \tag{46}$$

Equation (46) should be used instead of the wrong expression given by Flammer [his equation (4.6.14)]. This error in the expression of $R_{mn}^{(1)\prime}(-ic,0)$ further resulted in an uncorrected expression of $R_{mn}^{(2)}(-ic,0)$ [see Eq. (4.6.15b)of Ref. [2]]. The correct expression should then be

$$R_{mn}^{(2)}(-ic,i0)=-\frac{1}{cR_{mn}^{(1)\prime}(-ic,i0)}=\frac{i^{n-m+1}(2m+3)\sum_{r=1}^{\infty} d_r^{mn}(-ic)\dfrac{(2m+r)!}{r!}}{2^m m!c^{m+2}d_1^{mn}(-ic)}. \tag{47}$$

This can also be proved using the direct reduction. Obviously, we have the following relations:

$$j_n(z)\big|_{z\to 0}=\frac{2^n n!z^n}{(2n+1)!},\quad \left.\frac{dj_n(z)}{dz}\right|_{z\to 0}=\frac{2^n n!nz^{n-1}}{(2n+1)!}. \tag{48}$$

Since

$$I(c,\xi)=\frac{d}{d\xi}\left[\left(\frac{\xi^2-1}{\xi^2}\right)^{m/2} j_{m+r}(c\xi)\right]$$
$$=m\frac{(\xi^2-1)^{m/2-1}}{\xi^{m+1}}j_{m+r}(c\xi)+\frac{(\xi^2-1)^{m/2}}{\xi^m}cj'_{m+r}(c\xi); \tag{49}$$

thus we have

$$I(-ic,i\xi)\big|_{\xi\to 0}=im\frac{2^{m+1}(m+1)!}{(2m+3)!}(ic)^{m+1}\delta_{r1}$$
$$+c\frac{2^{m+1}(m+1)!}{(2m+3)!}(m+1)(ic)^m\delta_{r1}$$
$$=-i\frac{2^{m+1}c^{m+1}(m+1)!}{(2m+3)!}\delta_{r1}, \tag{50}$$

where δ_{r1} ($=1$ for $r=1$ and 0 otherwise) denotes the Kronecker delta. Taking the derivative of the radial spheroidal function of the first kind and assuming $\xi=0$, we have

$$R_{mn}^{(1)'}(-ic,i0)$$
$$=\frac{\sum_{r=1}^{\infty} i^{n-m-1}d_r^{mn}(-ic)\frac{(2m+r)!}{r!}iI(-ic,i0)}{\sum_{r=1}^{\infty} d_r^{mn}(-ic)\frac{(2m+r)!}{r!}}. \tag{51}$$

Substituting $I(-ic,i0)$ in Eq. (50) into Eq. (51), we can obtain the same form as Eq. (46).

In adopting MacPhie and Do-Nhat's approach to establishing the accuracy, the oblate Wronskian test value (44b) is computed using the same parameters provided by Flammer, i.e., $m=0,1$, and 2, $n=0,1,2$, and 3, and $c=0.2,0.5,0.8,1.0,1.5,2.0$, and 2.5. The Wronskian test value shows that a comparatively much higher accuracy has been achieved by us with our MATHEMATICA package. Good agreement between the theoretical Wronskian values and our computed Wronskian values are found to full precision accuracy. In addition, it also shows the inaccuracy in Flammer's values, which was already pointed out by MacPhie and Do-Nhat for the prolate case.

X. CONCLUDING REMARKS

In this paper, we reviewed various methods employed to evaluate prolate (or oblate) angular spheroidal harmonics of the first and second kinds and prolate (or oblate) radial spheroidal harmonics of the first to the fourth kinds, as well as their first-order derivatives and their eigenvalues. Based on this comparative study of the various methodologies, an efficient algorithm for numerically computing these functions and eigenvalues is developed with the widely accepted MATHEMATICA package.

First, an exact method, solving the continued fraction equations, is adopted in the numerical implementation; the algorithm developed is therefore very accurate, quite fast, and very efficient for computing eigenvalues λ_{mn} of these spheroidal functions with the complex argument c (where actually the dielectric medium is assumed to be lossy material). With existing computer facilities, it is found that the current algorithm employing the fractional function is more efficient and accurate, as compared with others available. As the argument c becomes very large (say $1000+i500$), quite a high oscillation is observed from the functional plot of the equation. Therefore, the technique for solving the continued fraction equations actually fails. To overcome this problem, the second algorithm was developed subsequently.

Second, the intermediate parameters d_r^{mn} (where r varies with a step size 2 from $-2m+\ell$ to ∞ and $\ell=0$ for even $n-m$ and 1 for odd $n-m$) and $d_{\rho|r}^{mn}$ (where r varies with a step size 2 from $2m+2-\ell$ to ∞) are computed with great care after an estimation of the eigenvalues λ_{mn}. The evaluated data published by Flammer in his appendices [2] have been considered as referenced results for comparison; those data have also been reproduced in a very popular and widely used handbook of mathematical functions by Abramowitz and Stegun [22], with permission from Flammer. Therefore, Flammer's book [2] has been regarded as a classic text for the spheroidal functions. It is found, however, that the computed results of coefficients $d_{\rho|r}^{mn}$, and therefore some of the radial spheroidal harmonics of the second kind $R_{mn}^{(2)}(\bigcirc)$ and their derivatives $R_{mn}^{(2)'}(\bigcirc)$ [where $(\bigcirc)=(c,\xi)$ for prolate spheroidal functions, and $(-ic,i\xi)$ for oblate spheroidal functions] are quite inaccurate, e.g., $R_{00}^{(2)}(5,1.077)$ has a relative error of 9.79% while its derivative $R_{00}^{(2)'}(5,1.077)$ has a relative error of 37.57%.

Third, it is found that the existing personal computing facilities are capable of evaluating the angular and radial spheroidal harmonics of the first kind $S_{mn}(c,\eta)$ [or $S_{mn}(-ic,\eta)$] and $R_{mn}^{(1)}(c,\xi)$ [or $R_{mn}^{(1)}(-ic,i\xi)$] in terms of the series of the associated Legendre functions $P_n^m(\eta)$ and the spherical Bessel functions $j_n(\xi)$. However, for angular and radial spheroidal harmonics of the second kind $S_{mn}^{(2)}\times(c,\eta)$ [or $S_{mn}^{(2)}(-ic,\eta)$] and $R_{mn}^{(2)}(c,\xi)$ [or $R_{mn}^{(2)}(-ic,i\xi)$], computation in terms of the associated Legendre functions $P_n^m(\eta)$ and the intermediate parameters d_r^{mn} and $d_{\rho|r}^{mn}$ is highly recommended.

Finally, the values of the oblate spheroidal wave functions of the second kind provided by Flammer [2] were found to be inaccurate. In view of this inaccuracy, the authors of this paper recalculated oblate radial functions of the second kind and their derivatives using the same parameters. The values obtained were then verified by using Wronskian test values. Also, various values provided by the programs attached in

Refs. [20] and [8] were also verified. It is found that the existing programs are quite useful to a certain extent in meeting the requirement of accuracy, but a few discrepancies are still found, for example, the values produced in Ref. [25]. Using Wronskian test values, it is shown that the algorithm developed in work with the MATHEMATICA package achieves the best accuracy to a full precision. While the other programs cannot be used to compute the spheroidal harmonics in lossy media, this algorithm can be utilized instead. Also, the algorithm works very well for both small and very large c's.

Some typographical errors in Ref. [2] should also be pointed out. The following list indicates the corrections to those errors to our awareness: (i) There is a missing minus sign in Eq. (3.1.7) of Ref. [2]. (ii) The second factorial $[(n-m)/2]!$ in Eq. (4.2.2a) of Ref. [2] should be read as $[(n+m)/2]!$ (iii) The denominator $\kappa^{(2)}(-ic)$ in Eq. (4.2.7) of Ref. [2] should be read as $\kappa^{(2)}_{mn}(-ic)$. (iv) The term ${}^{\cos}_{\sin}\cos(m+1)\phi$ of $M^{(+(i)}_{\substack{e\\o}m+1,n\phi}$ should be ${}^{\cos}_{\sin}(m+1)\phi$. (v) The correct form of the radial function $R^{(2)}_{03,}$ in Table 103 of Ref. [2] should be $R^{(2)}_{02}$. (vi) Equations (4.6.14) and (4.6.15b) of Ref. [2] should be corrected to our equations (46) and (47).

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